Topic 5 – Binomial Random Variables

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A Bernoulli trial is an experiment with two possible outcomes: success or failure.

Suppose success occurs with probability p and failure with probability $1-p$.

<u>Ex:</u> Experiment = flipping a coin

success = heads $\leftarrow$ $p = 1/2$

failure = tails $\leftarrow$ $1-p = 1/2$

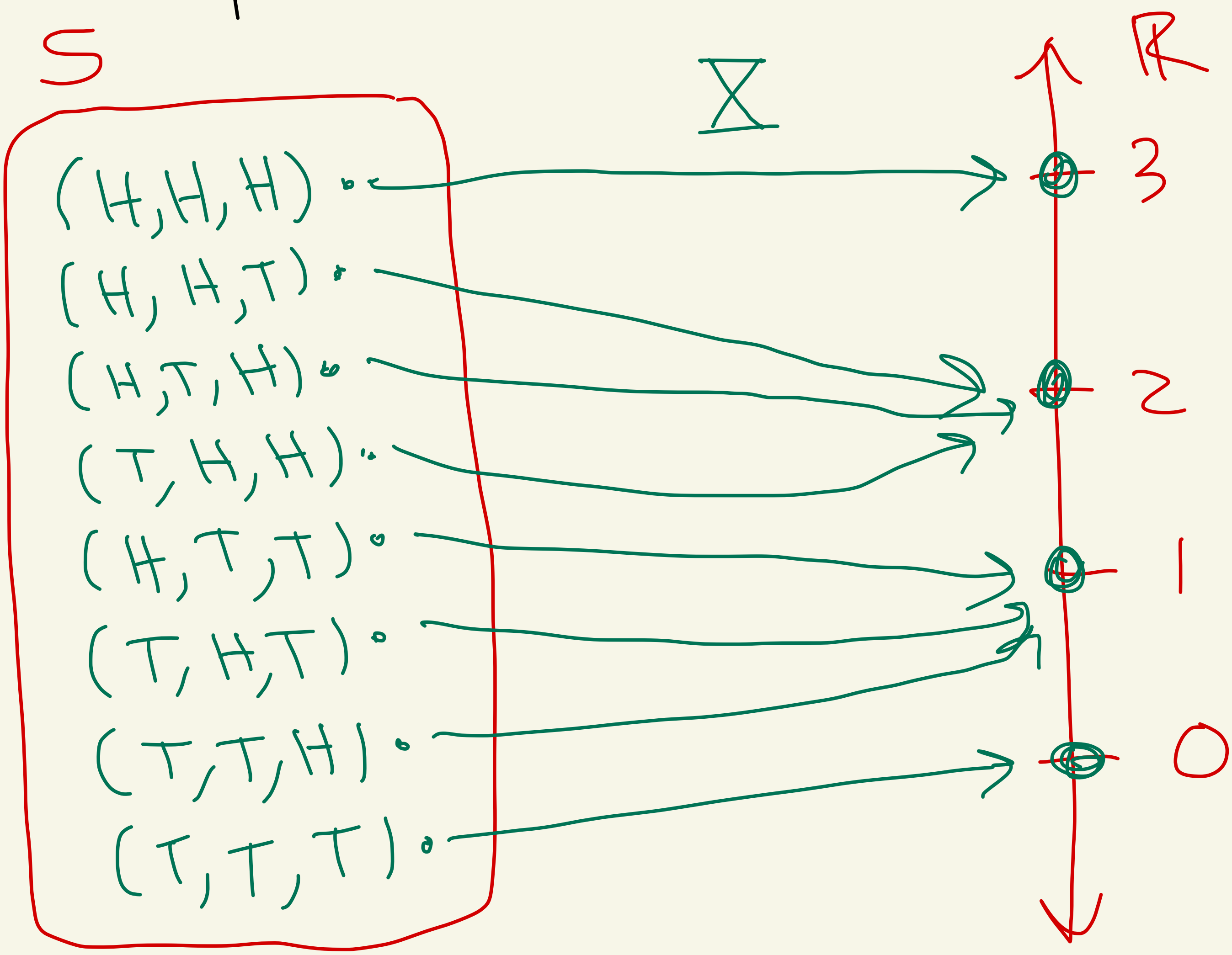
Ex: Experiment = rolling two 6-sided die

success = sum of dice is 7 ← $p = 6/36$

failure = sum of dice is not 7 ← $1-p$
$= 30/36$

Now suppose that n Bernoulli trials, each with success p, are performed independently. Let X be the number of successes. Then X is called a binomial random variable with parameters n and p.

Ex: Suppose the Bernoulli trial is flipping a coin and success is heads with probability $p = 1/2$. Let's repeat this experiment $n = 3$ times and let X be the number of heads that occur. Then X is a binomial random variable with parameters $n = 3$, $p = 1/2$.



Theorem: Let X be a binomial random variable with parameters n and p. Then,

$$P(X=k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k=0,1,2,\dots,n \\ 0 & \text{otherwise} \end{cases}$$

proof: Let k be $0,1,2,\dots$ or n.

Let's calculate $P(X=k)$.

How many ways can you get exactly k successes in n trials?

Ex: $n=4$, $k=2$

$\underline{s}\ \ \underline{s}\ \ \underline{f}\ \ \underline{f} \leftarrow \binom{4}{2}=6$

pick 2 spots where the successes go

ssff sffs fsfs

sfsf fssf ffss

For general n and k, there are $\binom{n}{k}$ ways you can get exactly k successes in n trials.

Each of these sequences has probability $\underbrace{p^k}_{k \text{ successes}}\underbrace{(1-p)^{n-k}}_{n-k \text{ failures}}$

because of independence.

Ex:

$$\underset{\uparrow}{\underline{s}} \quad \underset{\uparrow}{\underline{s}} \quad \underset{\uparrow}{\underline{f}} \quad \underset{\uparrow}{\underline{f}}$$

$$p \cdot p \cdot (1-p) \cdot (1-p) = p^2 (1-p)^{4-2}$$

Thus, $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$.

If $k \neq 0, 1, \ldots, n$, then X can never equal k, so $P(X = k) = 0$. $\blacksquare$

<u>Ex:</u> Suppose we flip a coin 3 times. What is the probability that exactly 2 heads occur?

Let $n = 3$, success = head, $p = 1/2$, X = # of heads in 3 flips.

$$P(X=2) = \binom{\overset{\text{\# flips}}{n}}{\underset{\text{\# successes}}{2}} p^2 (1-p)^{3-2}$$

$$= \binom{3}{2} \cdot \underbrace{\left(\frac{1}{2}\right)^2}_{\text{2 successes}} \underbrace{\left(1 - \frac{1}{2}\right)^1}_{\text{1 failure}}$$

or heads   or tails

$$= 3\cdot\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right) = \frac{3}{8}$$

<u>Ex:</u> What if flip a coin 100 times and you want the probability of exactly 48 heads occuring?

$n = 100$ ← # flips

$k = 48$ ← # successes/heads

$p = 1/2$ ← probability of one success

$1-p = 1 - 1/2 = 1/2$ ← probability of one failure

X = # successes/heads

$$P(X=48) = \binom{100}{48} \cdot \left(\frac{1}{2}\right)^{48} \left(1-\frac{1}{2}\right)^{100-48}$$

$$= \binom{100}{48} \cdot \left(\frac{1}{2}\right)^{48} \cdot \left(\frac{1}{2}\right)^{52}$$

$$= \binom{100}{48} \cdot \frac{1}{2^{100}}$$

$$= \frac{93,206,558,875,049,876,949,581,681,100}{1,267,650,600,228,229,401,496,703,205,376}$$

$$\approx 0.073527 \approx \boxed{7.35\%}$$

<u>Ex:</u> Suppose we flip a coin 20 times. What is the probability of getting between 10 and 12 heads? (ie 10, 11, or 12 heads).

$n = 20 \leftarrow$ # flips

$p = 1/2 \leftarrow$ probability of success/heads

$1-p = 1/2 \leftarrow$ probability of failure/tails

X = # of successes/heads

$$P(10 \le X \le 12)$$

$$= P(X=10) + P(X=11) + P(X=12)$$

$$= \binom{20}{10} \cdot \underbrace{\left(\frac{1}{2}\right)^{10}}_{\text{success}} \underbrace{\left(\frac{1}{2}\right)^{10}}_{\text{failure}} + \binom{20}{11} \underbrace{\left(\frac{1}{2}\right)^{11}}_{\text{success}} \underbrace{\left(\frac{1}{2}\right)^{9}}_{\text{failure}}$$

$$+ \binom{20}{12} \underbrace{\left(\frac{1}{2}\right)^{12}}_{\text{success}} \underbrace{\left(\frac{1}{2}\right)^{8}}_{\text{failure}}$$

$$= \frac{\binom{20}{10} + \binom{20}{11} + \binom{20}{12}}{2^{20}}$$

$$= \frac{184{,}756 + 167{,}960 + 125{,}970}{1{,}048{,}576}$$

$$\approx 0.456511\ldots \approx \boxed{45.65\%}$$

<u>Ex:</u> Suppose we roll two 6-sided dice 20 times. Suppose we say that a sum of 7 or 11 on the die is a success, and any other sum is a failure.

Let X = # of successes.

What is $P(X=12)$?

$n = 20$

$$p = \underbrace{\frac{6}{36}}_{\text{sum is } 7} + \underbrace{\frac{2}{36}}_{\text{sum is } 11} = \frac{8}{36} \leftarrow \text{probability of success}$$

$$1-p = 1 - \frac{8}{36} = \frac{28}{36} \leftarrow \text{probability of failure}$$

$$P(X=12) = \binom{20}{12} \cdot \underbrace{\left(\frac{8}{36}\right)^{12}}_{12 \text{ successes}} \cdot \underbrace{\left(\frac{28}{36}\right)^{8}}_{8 \text{ failures}}$$

$$= \binom{20}{12} \cdot \left(\frac{2}{9}\right)^{12} \cdot \left(\frac{7}{9}\right)^{8}$$

$$= \frac{\binom{20}{12} \cdot 2^{12} \cdot 7^{8}}{9^{20}}$$

$$= \frac{(125,970)(4096)(5,764,801)}{12,157,665,459,056,928,801}$$

$$\approx 0.000244659 \approx \boxed{0.024\%}$$

Let's see what the expected value of a binomial random variable is.

Intuition: Say we flip a coin 100 times then we would expect the average number of heads to be $n \cdot p = 100 \cdot \frac{1}{2} = 50$

Theorem: Let X be a binomial random variable with parameters n and p. Then $E[X] = np$.

proof:

$$E[X] = \sum_{k=0}^{n} k \cdot P(X = k)$$

$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

k=0 makes the term equal 0

$$= \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^{i} (1-p)^{(n-1)-i}$$

$\boxed{i = k-1}$

$$= np \left[p + (1-p) \right]^{n-1}$$

Binomial thm:

$$(x+y)^{l} = \sum_{i=0}^{l} \binom{l}{i} x^{i} y^{l-i}$$

$l = n-1, \; x = p, \; y = 1-p$

$$= np \left[1 \right]^{n-1}$$

$$= np$$

$\blacksquare$